GENERALIZATIONS OF THE DERHAM COMPLEX WITH APPLICATIONS TO DUALITY THEORY AND THE COHOMOLOGY OF SINGULAR VARIETIES*

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Given an algebraic variety or an analytic space X over the complex numbers one has the natural problem of expressing the topological invariants of X (i.e., if X is algebraic, we mean the topological invariants of the associated analytic space) in terms of the analytic or algebraic structure. For example, if X is nonsingular then the hypercohomology of the deRham complex $\Omega_X^{\cdot}$ of X is $H^{\cdot}(X, \mathbf{C})$, the classical (Čech or singular) cohomology of X. (Hypercohomology groups of a complex of sheaves $A^{\cdot}$ are denoted by $\mathbf{H}^{\cdot}(A^{\cdot})$ and may be calculated [6] as the cohomology groups of the complex of global sections $\Gamma(E^{\cdot})$ where $E^{\cdot}$ is any complex of flasque (or fine) sheaves which "resolves" $A^{\cdot}$, in the sense that there is a map of complexes $A^{\cdot} \to E^{\cdot}$ which induces an isomorphism of cohomology sheaves. For example if $\Omega_X^{\cdot}$ denotes the complex of holomorphic differential forms then it may be resolved (X nonsingular analytic) by the complex $E_X^{\cdot}$ of C^{∞} differential forms and the cohomology of $\Gamma(E_X^{\cdot})$ is $H(X, \mathbf{C})$ by deRham's Theorem. The analogous theorem for X nonsingular algebraic is due to Grothendieck [4] and employs resolution of singularities.) When X is singular, the deRham complex may no longer calculate $H(X, \mathbf{C})$, [9]. The difficulties arise from the failure of the "Poincaré lemma," i.e., from the impossibility of smoothly contracting small neighborhoods of singular points. It has been shown by Deligne (unpublished) and independently by Herrera-Lieberman [7], that if one imbeds X in a nonsingular space Z, then the deRham complex on the formal neighborhood of X in Z calculates the cohomology of X correctly. (Intuitively, the formal neighborhood, $\hat{X}$, is like a tubular neighborhood of X in Z, and is "homotopy equivalent" to X, while on the other hand $\hat{X}$ is "nonsingular" so that its deRham and topological cohomologies coincide.)

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The proofs of the above theorems all require resolution of singularities. Recently we have discovered a method of calculating cohomology of complete singular varieties X intrinsically (i.e., without imbedding X in a nonsingular space) and the construction and proof does not require resolution of singularities. We study a system of complexes $B_n^{\cdot}$ generalizing the deRham complex (which is $B_1^{\cdot}$). The complex $B_n^{\cdot}$ is a complex of coherent sheaves, the differentials are differential operators of order n. When X is nonsingular the complexes $B_n^{\cdot}$ are each elliptic and each calculate the cohomology of X. In the presence of singularities the (inverse) limit of the B_n cohomology is the correct cohomology of X.

The study of the B_n is intimately related to Grothendieck's theory of the stratifying topos ([5], §5) and the assertion that $\varprojlim(B_n)$ calculates $H(X, \mathbf{C})$ yields an affirmative answer to conjecture 5.1 of [5].

The initial study of the B_n was motivated by the desire to produce a good duality theory for differential operators. As is well known, if $E^{\cdot}$ is a complex of locally free sheaves with differential operators $D_i: E^i \to E^{i+1}$, then one can introduce adjoint differentials $D_i^*: (E^{i+1})^* \otimes \Omega^m \to (E^i)^* \otimes \Omega^m$ (Ω^m denotes the highest degree forms). The adjoint is obtained transcendentally (see for example [1]) by employing global integration on X to characterize the adjoint as a map between currents and then noting that the operator so defined actually preserves the algebraic (or analytic) sheaves. What is an algebraic characterization and construction of the adjoint? How should the theory extend to complexes in which the sheaves are only assumed to be coherent? Why does the sheaf Ω^m figure in the definition of the adjoint? From the global differentiable point of view it is convenient since it gives objects which can be integrated. We see below that the choice of Ω^m is essential even from a local point of view. A duality theory for $E^{\cdot}$ with differential operators of order 1 was given in [7], by noting that every such $E^{\cdot}$ had a canonical structure of graded module over the ring $\Omega^{\cdot}$. Then

$$\mathrm{Hom}_{\mathcal{O}}(E^{\cdot}, \Omega^m) \xrightarrow{\sim} \mathrm{Hom}_{\Omega^{\cdot}}(E^{\cdot}, \mathrm{Hom}_{\mathcal{O}}(\Omega^{\cdot}, \Omega^m)) \xrightarrow{\sim} \mathrm{Hom}_{\Omega^{\cdot}}(E^{\cdot}, \Omega^{\cdot})$$

by using the change of rings formula and the canonical isomorphism $\mathrm{Hom}_{\mathcal{O}}(\Omega^{\cdot}, \Omega^m) \xrightarrow{\sim} \Omega^{\cdot}$. The natural differential D^* in $\mathrm{Hom}_{\Omega^{\cdot}}(E^{\cdot}, \Omega^{\cdot})$ given by

$$(D^*\Phi)(\alpha) = (-1)^p \Phi(D_p(\alpha)) + (-1)^{p+1} d(\Phi(\alpha)); \ \alpha \in E^p$$

is the adjoint. The complexes B_n generalize this theory: they are differential graded rings and every $E^{\cdot}$ with differentials of order $\leqq n$ is canonically a module over B_n. Again

$$\mathrm{Hom}_{\mathcal{O}}(E^{\cdot}, \Omega^m) \xrightarrow{\sim} \mathrm{Hom}_{B_n}(E^{\cdot}, \mathrm{Hom}_{\mathcal{O}}(B_n, \Omega^m))$$

and although $\mathrm{Hom}_{\mathcal{O}}(B_n^{\cdot}, \Omega^m) \not\xrightarrow{\sim} B_n$, it has a canonical differential d and the adjoint in $\mathrm{Hom}(E, \Omega^m)$ is given by the formula above.

§1. *Amitsur Cohomology*

The essential problem in calculating the cohomology of an algebraic variety X is that the Čech cohomology with $\mathbf{C}$ coefficients (Zariski topology) is *not* the (standard) Čech cohomology of X_{an}. Thus an algebraic complex cannot be shown to calculate standard cohomology by proving a "Poincaré lemma," i.e., by showing that it resolves the sheaf $\mathbf{C}$, since it would then calculate Zariski-Čech cohomology. If, however, X is *complete* and $E^{\cdot}$ is a complex of algebraic *coherent* sheaves and $E_{an}^{\cdot}$ is the associated complex of analytic sheaves, then by GAGA [*10*], $E^{\cdot}$ and $E_{an}^{\cdot}$ have the same hyper-cohomology, and this will be $H^{\cdot}(X_{an}, \mathbf{C})$ if E_{an} satisfies a Poincaré lemma. Thus in the *complete* case the problem is purely an *analytic* one, namely to construct a canonical *coherent* complex $E_{an}^{\cdot}$ which satisfies a "Poincaré lemma." We point out first two highly canonical *non-coherent* complexes, the Amitsur and the Alexander-Spanier. The complexes $B_n^{\cdot}$ will then be constructed as coherent approximations to these complexes. Both the Amitsur and Alexander-Spanier constructions provide canonical resolutions of $\mathbf{C}$, hence can be used to calculate cohomology in the analytic, differentiable or topological categories. (In the algebraic category one calculates Zariski cohomology which is "wrong.")

Given $(X, \mathcal{O}_X)$ a ringed space over $\mathbf{C}$ consider the co-simplicial complex of sheaves of rings, $A^i = \otimes_{\mathbf{C}}^{i+1} \mathcal{O}_X$ with face maps

$$d_j(f_0 \otimes \cdots \otimes f_i) = f_0 \otimes \cdots \otimes f_{j-1} \otimes 1 \otimes f_j \otimes \cdots \otimes f_{i+1}$$

and degeneracy operators

$$s_j(f_0 \otimes \cdots \otimes f_{i+1}) = f_0 \otimes \cdots f_{j-1} \otimes f_j f_{j+1} \otimes f_{j+2} \cdots \otimes f_{i+1} \, .$$

The associated complex of sheaves with boundary, $d = \sum(-1)^j d_j$, is called the Amitsur complex. The *normalized Amitsur complex* $B^{\cdot}$ is the subcomplex of $A^{\cdot}$

$$B^{\cdot} = \bigcap_i \ker(s_i)$$

or equivalently under the natural projection,

$$B^{\cdot} \xrightarrow{\sim} A^{\cdot} / \sum_{i \geqq 1} \mathrm{im}\,(d_i).$$

One has the well-known

Theorem. *$A^{\cdot}$ and $B^{\cdot}$ are resolutions of the constant sheaf* $\mathbf{C}$.

Proof. Given any $x \in X$ the Amitsur homotopy

$$H_x(f_0 \otimes \cdots \otimes f_i) = f_0(x) f_1 \otimes \cdots \otimes f_i$$

preserves $B^{\cdot}$ and gives a homotopy for d.

The global sections of $A^{\cdot}$ therefore calculate cohomology of X (paracompact) in the differentiable or topological categories.

The Alexander-Spanier construction which is *not* used in the sequel is described briefly for comparison. Consider the simplicial system of spaces $X_i =$ product of $i+1$ copies of X with face maps $d_j(x_0, \cdots, x_i) = (x_0, \cdots, \hat{x}_j, \cdots, x_i)$ and degeneracy $s_j(x_0, \cdots, x_i) = (x_0, \cdots, x_{j-1}, x_j, x_j, \cdots, x_i)$. Denote by $(AS)^i$ the sheaf of rings on X given by the sheaf theoretic restriction of $\mathcal{O}_{X_i}$ to the multidiagonal, where $\mathcal{O}_{X_i}$ is the sheaf of (resp.: all, continuous, differentiable, analytic) $\mathbf{C}$-valued functions on X_i. The sheaves $(AS)^i$ form a co-simplicial system on X with face and degeneracy induced by d_j and s_j and the associated complex is called the complex of (resp.: standard, continuous, differentiable, analytic) Alexander-Spanier cochains. Each of them provides a resolution of $\mathbf{C}$. The Amitsur complex defines a subcomplex of $(AS)^{\cdot}$ via

$$(f_0 \otimes \cdots \otimes f_i)(x_0, \cdots, x_i) = f_0(x) f_1(x_1) \cdots f_i(x_i)$$

The Amitsur complex $A^{\cdot}$ also has a natural graded ring structure:

$$(f_0 \otimes \cdots \otimes f_i) \times (g_0 \otimes \cdots \otimes g_j) = f_0 \otimes \cdots \otimes f_i g_0 \otimes g_1 \otimes \cdots \otimes g_j$$

and the normalized complex $B^{\cdot}$ is a graded subring.

An alternative description of these rings is exceedingly useful for computation. Indeed, $A^{\cdot}$ is a "tensor algebra" of $A^1 = \mathcal{O} \otimes_{\mathbf{C}} \mathcal{O}$ over $A^0 = \mathcal{O}$, i.e., the multiplications $A^0 \otimes A^i \to A^i$ and $A^i \otimes A^0 \to A^i$ yield natural left and right $\mathcal{O}$ module structures on A^i, and multiplication induces an isomorphism $A^i \otimes_{\mathcal{O}} A^j \xrightarrow{\sim} A^{i+j}$ where we tensorize with respect to the *right* $\mathcal{O}$ structure of A^i and the *left* $\mathcal{O}$ structure of A^j. Similarly $B^{\cdot}$ is the tensor algebra of $B^1 = \ker(\mathcal{O} \otimes_{\mathbf{C}} \mathcal{O} \xrightarrow{s_0} \mathcal{O})$ over $B^0 = \mathcal{O}$.

Notations: We shall employ the symbol $\otimes$ for multiplication in the Amitsur ring with the conventions $f \otimes \phi = f \cdot \phi$ and $\phi \otimes f = \phi \cdot f$ for $f \in \mathcal{O}$. The tensor over $\mathbf{C}$ which has previously been denoted $\otimes$ will be written $\otimes_{\mathbf{C}}$. The individual A^i and B^i are themselves commutative rings and products therein will be expressed by juxtaposition, i.e., for f, $g \in \mathcal{O}$ we have df and $dg \in A^1$ and $(df)(dg) \in A^1$ while $df \otimes dg \in A^2$. Exponential notation is reserved for the commutative products, i.e., $(df)^2 = (df)(df)$ or more generally given $f_1, \cdots, f_n \in \mathcal{O}$ and $I = (i_1, \cdots, i_n)$ a vector of non-negative integers, $f^I = f^{i_1} \cdots f^{i_n}$ and $(df)^I = (df)^{i_1} \cdots (df)^{i_n} \in A^1$. Given I we set $|I| = i_1 + \cdots + i_n$, and $I! = i_1!, i_2! \cdots i_n!$ and $\binom{I}{J} = I!/J!(I-J)!$. The relationship of d to the various products is given by Leibniz rules. The first of these

$$\text{(L1)} \qquad d(\alpha \otimes \beta) = d\alpha \otimes \beta + (-1)^p \alpha \otimes d\beta \qquad \alpha \in A^p$$

shows that the product in A descends to give a product in cohomology, namely cup product.

Noting that any element of B^1 may be expressed as a finite sum of elements of the form $f \cdot dg$ with $f, g \in \mathcal{O}$ and that $B^{\cdot}$ is a tensor algebra we see that elements of B^p are sums of monomials $f dg_1 \otimes dg_2 \otimes \cdots \otimes dg_p$. Then d is simply expressed by

$$d(f dg_1 \otimes \cdots \otimes dg_p) = df \otimes dg_1 \otimes \cdots \otimes dg_p$$

in view of (L1) and $d^2 = 0$.

The relationship of d to the other products is expressed by

$$\text{(L2)} \qquad d(f^I) = \sum_{0 \lneq J} \binom{I}{J} f^{I-J} (df)^J$$

or consequently if $P(x_1, \cdots, x_n) \in \mathbf{C}[x_1, \cdots, x_n]$ is a polynomial then

$$\text{(L2)}' \qquad d(P(f_1, \cdots, f_n)) = \sum_{0 \lneq J} \frac{1}{J!} \frac{\partial^{|J|} P}{\partial x^J} \bigg|_{f_1 \cdots f_n} (df)^J$$

It is frequently useful to invert the rule (L2) to express $(df)^I$ in terms of $d(f^J)$ by the formula

$$\text{(L3)} \qquad (df)^I = \sum_J \binom{I}{J} (-f)^J d(f^{I-J})$$

Both formula (L2) and (L3) follow easily by recalling $d = d_0 - d_1$ and observing that d_0 and d_1 are ring homomorphisms.

Finally we have

$$\text{(L4)} \qquad d((df)^I) = - \sum_{0 \lneq J \lneq I} \binom{I}{J} (df)^J \otimes (df)^{I-J}$$

which is established as follows:

$$\begin{aligned} d((df)^I) &= d_0((df)^I) - d_1((df)^I) + d_2((df)^I) \\ &= (d_0(df))^I - (d_1(df))^I + (d_2(df))^I \end{aligned}$$

Since d_j are ring homomorphisms. But since $d(df) = 0$ or $d_1(df) = d_0(df) + d_2(df)$, we find (L4) by substituting in the middle terms and multiplying out, noting that $(d_0(df))^A \cdot (d_2(df))^B = (df)^B \otimes (df)^A$.

§2. *nth order deRham theory*

If X is a differentiable, analytic variety then one has a canonical homomorphism of differential graded rings of $B^\cdot$ onto the deRham complex $\Omega^\cdot$, sending $f dg_1 \otimes \cdots \otimes dg_p$ to $f\, dg_1 \wedge \cdots \wedge dg_p$. The kernel of this map is readily identified as the differential ideal of $B^\cdot$ generated by $B^1 \cdot B^1$, i.e., locally, by elements of the form $df \cdot dg \in B^1$. (The skew symmetry of $\Omega^\cdot$ arises precisely from annihilating

$$d(df \cdot dg) = -df \otimes dg - dg \otimes df, \text{ [cf. (L4)].)}$$

Let I_{n+1} be the ideal $B^1 \cdot B^1 \cdot \cdots \cdot B^1$, $((n+1)$ times), of B^1.

Definition. The nth order deRham complex $B^\cdot_n$ is the quotient of the normalized Amitsur complex $B^\cdot$ by the *differential ideal* generated by $I_{n+1} \subseteq B^1$, i.e., locally generated by elements of the form $(df)^J$, $|J| \geqq n+1$.

The nth order Amitsur complex $A^\cdot_n$ is the quotient of $A^\cdot$ by the differential ideal (of $A^\cdot$) generated by I_{n+1}.

Thus $B^0_n = \mathcal{O}_X$ and $B^1_n =$ normalized n-jets of X with $d\colon \mathcal{O}_X \to B^1_n$ being the map which assigns to a function its n-jet minus the constant term. If X is a manifold with local coordinates $x_1, \cdots, x_r$ then $(dx)^I$ $|I| \leqq n$ give a local $\mathcal{O}_X$ basis for B^1_n and

$$df = \sum \frac{1}{I\,!} \frac{\partial^{|I|} f}{\partial (x)^I} \cdot (dx)^I.$$

In general let J_{n+1} denote the *ideal* of $B^\cdot$ generated by $I_{n+1} \subset B^1$ so that $B^\cdot_n = B^\cdot / J_{n+1} + d(J_{n+1})$. Note that $B^\cdot / J_{n+1}$ is simply the tensor algebra of B^1_n over $\mathcal{O}_X$ and that $B^\cdot_n$ is obtained from this tensor algebra by annihilating the image of $d(J_{n+1})$ modulo J_{n+1}. Precisely, one must annihilate the ideal of the tensor algebra generated by elements of degree 2 of the form $d((df)^I)$ $|I| \geqq n+1$. Thus in terms of local coordinates B^p_r is generated as a left $\mathcal{O}$ module by $(dx)^{I_1} \otimes \cdots \otimes (dx)^{I_p}$ with $|I_j| \leqq n$, subject to the symmetries

$$\sum_{\substack{|J|\leqq n \\ |I-J|\leqq n}} \binom{I}{J} (dx)^J \otimes (dx)^{I-J} = 0 \quad n < |I| \leqq 2n\,.$$

For example in the case X is one dimensional with coordinate x, $B_2^{\cdot}$ is described as follows:

$B_2^0 = \mathcal{O}_X$ and B_2^p is a free $\mathcal{O}$ module of rank 2 with basis $dx \otimes \ldots \otimes dx$ and $(dx)^2 \otimes dx \otimes \cdots \otimes dx$.

Multiplication is $\otimes$ multiplication subject to

$$(dx)^2 \otimes (dx)^2 = 0$$

$$(dx) \otimes (dx)^2 = -(dx)^2 \otimes dx\,.$$

Care must be exercised in bringing functions through the $\otimes$, e.g.

$$dx \otimes g\,dx = dx \cdot g \otimes dx = g \cdot dx \otimes dx + \frac{dg}{dx}(dx)^2 \otimes dx$$

In general the sheaves B_n^p are always coherent, and when X is non-singular are locally free.

§3. *Cohomology of the* $B_n^{\cdot}$

Let $B_\infty^{\cdot} = \varprojlim B_n^{\cdot}$. If X is a manifold with local coordinates $x_1, \cdots, x_n$, local sections of $B_\infty^{\cdot}$ are uniquely represented in the form

$$\sum C_{IJ_1\cdots J_n} x^I (dx)^{J_1} \otimes \cdots \otimes (dx)^{J_n}$$

where $\sum C_{IJ\cdots J_n} x^I$ are power series, convergent on the coordinate poly-disc. $B_n = B_\infty / J_n + d(J_n)$ where J_n is the two sided ideal of B_∞ generated by the $(dx)^I$ with $|I| \geqq n+1$. In terms of local coordinates we have an explicit Poincaré homotopy operator for $B_\infty^{\cdot}$, namely,

$$P(x^I (dx)^{J_1} \otimes \ldots \otimes (dx)^{J_p})$$

$$= \frac{1}{|I| + |J_1| + \cdots + |J_p|} x^I \sum_{|J_k|=1} (-1)^k (dx)^{J_1} \otimes \ldots \otimes x^{J_k} \otimes \ldots \otimes (dx)^{J_p}$$

Notice that P preserves J_n, and from the formula $Pd + dP = 1$ it follows that P preserves dJ_n hence providing a homotopy for each $B_n^{\cdot}$. (In the differentiable category one can also construct a "P" for B_∞ and the B_n following lines similar to [2].) Thus in the non-singular case each of the B_n calculates the correct cohomology of X.

Theorem. *If X is a compact analytic space (or a complete algebraic variety), then $H(X,\mathbf{C})$ is calculable as the hypercohomology $\mathbf{H}(B_\infty) = \varprojlim \mathbf{H}(B_n)$ where B_n is the analytic (or algebraic) nth order deRham complex and $B_\infty = \varprojlim B_n$.*

Proof. The fact that $\varprojlim \mathbf{H} = \mathbf{H}(\varprojlim)$ follows by standard Mittag-Loeffler arguments (cf. [*6*]). The theorem for algebraic X follows by considering the associated analytic space X_{an} and noting that since the B_n are coherent, one has $\mathbf{H}(X, B_n^{\cdot}) \xrightarrow{\sim} \mathbf{H}(X_{an}, B_{n,an}^{\cdot})$ by applying GAGA, [*10*]. Thus the theorem is reduced to the assertion that $B_\infty^{\cdot}$ resolves $\mathbf{C}$ in the analytic case. This is a local question, hence we may assume that X is a closed subvariety of a polydisc U. Denoting by $B_n(U)$, $B_n(X)$ the global sections of the sheaves, we have the short exact sequences

$$0 \to \Sigma_n \to B_n(U) \to B_n(X) \to 0$$

and

$$0 \to \Sigma_\infty \to B_\infty(U) \to B_\infty(X) \to 0$$

where Σ_n (resp.) Σ_∞ is the kernel of the restriction map and $\Sigma_\infty = \varprojlim \Sigma_n$ (again using Mittag-Loeffler). The Poincaré homotopy P on U shows that $B_\infty(U)$ resolves $\mathbf{C}$, hence $B_\infty(X)$ resolves $\mathbf{C}$ if and only if Σ_∞ is acyclic. The problem is that P does not in general preserve Σ_∞. (One can check easily that if the equations defining X can be chosen to be homogeneous, then P does preserve Σ_∞ and all the Σ_n.)

The Amitsur homotopy H described earlier extends to provide a "formal" homotopy for B_∞. In terms of the coordinates $x_1 \cdots x_n$ on U,

$$H(\Sigma C_{IJ_1\dots J_p} x^I (dx)^{J_1} \otimes \cdots \otimes (dx)^{J_p}) = \Sigma C_{0J_1\dots J_p} x^{J_1} (dx)^{J_2} \otimes \cdots \otimes (dx)^{J_p}$$

where the right hand side is not in general convergent. However, H does preserve Σ_∞ (inducing the formal Amitsur homotopy for $B_\infty(X)$). Given $\gamma \in \Sigma_\infty$ with $d\gamma = 0$ one has $\gamma = d(H\gamma)$ which is only a formal solution. One may smooth this formal solution by the following kind of technque. Truncate $H\gamma$ as $H\gamma = H_k(\gamma) + R_k(\gamma)$ where $H_k\gamma$ is convergent and $R_k(\gamma) \in M^k \cdot \Sigma_\infty$ where M^k denotes formal series beginning with terms of order k. Note that $d(R_k\gamma)$ is convergent, being equal to $\gamma - d(H_k(\gamma))$.

Now $P(dR_k\gamma)$ is convergent and since $d(dR_k\gamma) = 0$ we see $d(PdR_k\gamma) = dR_k\gamma$. Thus if we set $\phi_k = H_k(\gamma) + PdR_k(\gamma)$ then ϕ_k is convergent and $d\phi_k = \gamma$. The ϕ_k so constructed not need be in $\Sigma(X)$ (the ideal of forms vanishing on X) but is in the ideal of forms vanishing on $X \cap k$th order neighborhood of zero. To actually show γ bounds in $\Sigma(X)$, one must coherently put together the information that γ bounds on $X \cap k$th neighborhood of every point. To achieve this one considers the relative deRham theory of $U \times U \xrightarrow{\pi_1} U$ replacing X by $U \times X \to U$ and individual points by the diagonal $U \to U \times U$. Using the relativized versions of P and H (whose restrictions to the fibers $x \times U$ yield the P and H based at x), one argues as above to show γ bounds.

§4. *Duality Theory*

Given any space X, the Amitsur complex is a sheaf of differential graded algebras which is characterized by the following universal property:

The category of sheaves of graded $A^{\cdot}$ modules coincides with the category "sequences of $\mathcal{O}_X$ modules M^i together with C linear maps $D: M^i \to M^{i+1}$." Given such a sequence one obtains the $A^{\cdot}$ module structure by

$$(a_0 \otimes \cdots \otimes a_i) \cdot (m) = a_0(D(a_1(D \cdots a_i m)) \cdots)$$

Note that the map $D: M^i \to M^{i-1}$ is simply left multiplication by the element $1 \otimes_C 1 \in A^1$, and we refer to this element as D. Note that an $A^{\cdot}$ module $M^{\cdot}$ gives rise to a complex of $\mathcal{O}$ modules if and only if $M^{\cdot}$ is annihilated by $D \otimes D \in A^2$, hence if and only if $M^{\cdot}$ is a graded module over the ring $C^{\cdot} = A^{\cdot}/(D \otimes D)$. The ring $C^{\cdot}$ has an alternative description in terms of the normalized Amitsur complex $B^{\cdot}$, namely $C^{\cdot} = B^{\cdot} \oplus B^{\cdot -1}$. D is as readily checked. $C^{\cdot}$ has two differentials, d (inherited from the Amitsur complex) and D (i.e., left multiplication by D). $B^{\cdot}$ is a subring of $C^{\cdot}$ and as preserved by d (but not D). In a similar manner, if A_n denotes the nth order Amitsur complex and $C_n = A_n/D \otimes D$ then the graded A_n (resp. C_n) modules are precisely the sequences (resp. complexes) of $\mathcal{O}_X$ modules M^i with $M^i \to M^{i+1}$ being a *differential* operator of order n.

When X is a compact differential manifold, and $\mathcal{O}$ is the "volume bundle" [*1*] (i.e., the sheaf of "formes d'espèce impair" of deRham [*8*]), then the assignment $M^{\cdot} \to \mathrm{Hom}_{\mathcal{O}}(M^{\cdot}, \mathcal{O})$ defines a contravariant functor from the Amitsur modules having locally free M^i to themselves. Indeed, every C linear map $M^i \to M^{i+1}$ is actually a differential operator of some finite order by the theorem of Peetre, and the adjoint $\mathrm{Hom}(M^{i+1}, \mathcal{O}) \to \mathrm{Hom}(M^i, \mathcal{O})$ defines the Amitsur module structure. This functor preserves the categories of C, A_n, and C_n modules.

We investigate this sort of duality phenomenon for the algebraic and analytic categories. Since the analog of Peetre's theorem appears to be unknown for the analytic category, and is definitely false in the algebraic category, we shall consider only Amitsur modules in which the maps are differential operators.

We assume throughout this section that X is non-singular of dimension m.

For V open in X denote by Mod_V the category nf $\mathcal{O}_V$ modules and by Ab_V the category of Abelian sheaves on V. Consider systems of contravariant additive left exact functors $F_V\colon \mathrm{Mod}_V \to \mathrm{Ab}_V$ which commute with the restriction functors $\mathrm{Mod}_V \to \mathrm{Mod}_U$ and $\mathrm{Ab}_V \to \mathrm{Ab}_U$ for $U \subseteq V$. Note that $F(M)$ has a natural $\mathcal{O}$ module structure and that there exists a canonical $\mathcal{O}$ homomorphism

$$F(M) \to \mathrm{Hom}_{\mathcal{O}}(M, F(\mathcal{O}))$$

which is an isomorphism if M is coherent (cf. [3], §4). Thus all candidates for duality theories are "essentially" of the form $M \rightsquigarrow \mathrm{Hom}_{\mathcal{O}}(M, L)$. For convenience we assume $F(M) = \mathrm{Hom}(M, L)$. We say F is *reflexive* if the natural map $\mathcal{O} \to F(F(\mathcal{O}))$ is an isomorphism. F is called *coherent* if it carries coherent modules to coherent modules, i.e., if and only if $L = F(\mathcal{O})$ is coherent. Every system of F's extends immediately to a functor from Amitsur modules to graded $\mathcal{O}_X$ modules, where we define $(F(E^{\cdot}))^i = F(E^{m-i})$, $m = \dim X$.

When the Amitsur module $E^{\cdot}$ has $\mathcal{O}$ linear maps $E^i \to E^{i+1}$ then $F(E^{\cdot})$ also has a natural Amitsur structure with $\mathcal{O}$ linear differentials. The F's will be said to define a *preduality theory* if for every Amitsur module $E^{\cdot}$ one can functorially prescribe an Amitsur module structure on $F(E^{\cdot})$ compatible with its graded $\mathcal{O}$ module structure and yielding the natural Amitsur structure when $E^{\cdot}$ is $\mathcal{O}$ linear (i.e., one can define adjoint differentials in $F(E^{\cdot})$ in a functorial way). The preduality theory is called *special* if F carries complexes (i.e., $C^{\cdot}$ modules) to complexes. (F is special if and only if the adjoint of a composition of $\mathbf{C}$ linear maps is the composition of the adjoints.) A duality theory is a reflexive, coherent, special preduality theory. We shall see below that $M \to \mathrm{Hom}_{\mathcal{O}}(M, R)$ defines a special preduality theory if and only if $\mathrm{Hom}_{\mathcal{O}}(\Omega^m, L)$ has an integrable connection.

Thus $M \to \mathrm{Hom}(M, L)$ is a duality theory if and only if $L \xrightarrow{\sim} \Omega^m \otimes_{\mathbf{C}} V$ where V is a local system (i.e., locally free $\mathbf{C}$ sheaf) of rank 1. Hence Ω^m is uniquely characterized as giving rise to the only "good" notion of duality.

The fact that $(\Omega^m)^* \otimes L$ has an integrable connection if L defines a special preduality theory follows from applying F to the deRham complex $\Omega^{\cdot}_X$ denoting $F(d)$ by d^*. Note first

$$F(\Omega^{\cdot})^p = \mathrm{Hom}(\Omega^{m-p}, L) \xrightarrow{\sim} \Omega^p \otimes \mathrm{Hom}(\Omega^m, L)$$

and that the differential $d^*\colon \Omega^p \otimes \mathrm{Hom}(\Omega^m, L) \to \Omega^{p-1} \otimes \mathrm{Hom}\,(\Omega^m, L)$ satisfies the rule

$$d^*(\phi \otimes h) = d\phi \otimes h + (-1)^{m-p}\phi \wedge d^*h \tag{1}$$

obtained by "dualizing" the relation between exterior multiplication and differentiation in $\Omega^{\cdot}$. Thus $F(\Omega^{\cdot})$ with the differential $(-1)^m d^*$ is the complex obtained by fixing an integrable connection on $\mathrm{Hom}(\Omega^m, L)$.

Conversely fixing an L and an integrable connection on $\mathrm{Hom}(\Omega^m, L)$, we can define the dual of the deRham complex by reversing the above. We seek to build a full preduality theory, i.e., to introduce an Amitsur module structure on $\mathrm{Hom}_{\mathcal{O}}(E^{\cdot}, L)$, (i.e., to define a C linear map $D_E^* \mathrm{Hom}(E^{\cdot}, L) \to \mathrm{Hom}_{\mathcal{O}}(E^{\cdot -1}, L)$) which is functorial in the Amitsur module $E^{\cdot}$.

Note the change of rings formula

$$\mathrm{Hom}_{\mathcal{O}}(E^{\cdot}, L) \xrightarrow{\sim} \mathrm{Hom}_{A^{\cdot}}(E^{\cdot}, \mathrm{Hom}_{\mathcal{O}}(A^{\cdot}, L))$$

where $\mathrm{Hom}_{\mathcal{O}}(A^{\cdot}, L)$ is viewed as a left $A^{\cdot}$ module via $A^{\cdot}$ acting by right multiplication on $A^{\cdot}$. In view of this formula we see that to functorially define D_E it is necessary and sufficient to define

$$D_A^*: \mathrm{Hom}_{\mathcal{O}}(A^{\cdot}, L) \to \mathrm{Hom}_{\mathcal{O}}(A^{\cdot -1}, L)$$

so that it is *linear* with respect to the left $A\cdot$ structure. But $A\cdot$ linearity forces the definition of $D_A^*: \mathrm{Hom}(A^i, L) \to \mathrm{Hom}(A^{i-1}, L)$ for all i once $D_A^*: \mathrm{Hom}(A^1, L) \to \mathrm{Hom}(A^0, L) = L$ is defined, (i.e., given $l \in \mathrm{Hom}(A^i, L)$ and $\alpha \in A^{i-1}$

$$(D^*l)(\alpha) = (D^*l)(1 \cdot \alpha) = (\alpha \cdot D^*l)(1) = D^*(\alpha \cdot l)(1) \text{ where } \alpha \cdot l \in \mathrm{Hom}(A^1, L).$$

Thus the preduality theory will be completely specified by defining $D_A^*: \mathrm{Hom}_{\mathcal{O}}(A^1, L) \to L$ which must be $\mathcal{O}$ linear with respect to $\mathcal{O}$ acting on A^1 from the *right*.

Note that the preceding remarks about $A\cdot$ are equally valid for the nth order Amitsur $A_n^{\cdot}$. Thus adjoints for nth order operators are determined by defining $D_n^* \mathrm{Hom}_{\mathcal{O}}(A_n^1, L) \to L$ (suitably $\mathcal{O}$ linear). We recall that $A_n^1 = J_n$ is the sheaf of nth order jets. Thus D_n^* will be the adjont of the "universal" nth order operator.

Observe that $\mathrm{Hom}_{\mathcal{O}}(A^1, L)$ is filtered by the subsheaves

$$\mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, L) \subseteq \mathrm{Hom}_{\mathcal{O}}(J_1, L) \subseteq \cdots \subseteq \mathrm{Hom}_{\mathcal{O}}(J_n, L) \subseteq \cdots$$

and that $D_A^*: \mathrm{Hom}_{\mathcal{O}}(\mathcal{O}, L) = L \to L$ must be the identity map in view of our requirement that the "adjoint" of an $\mathcal{O}$ linear map be its transpose. Moreover, $J_n = \mathcal{O} \oplus B_n^1$ as a left $\mathcal{O}$ module so that $D_A^*: \mathrm{Hom}(J_n, L) \to L$ is completely determined by $d_n^*: \mathrm{Hom}(B_n^1, L) \to L$. In particular the connection on $\mathrm{Hom}(\Omega^m, L)$ already determines $d^*: \mathrm{Hom}(B^1, L) = \mathrm{Hom}(\Omega^1, L) \to L$ and hence the entire first order duality theory. The required $\mathcal{O}$ linearity of D^* is a restatement of the connection rule (1) for d^*.

The nth order duality theory is now forced by "composition." More precisely, one has the natural injection

$$J_n \xrightarrow{\lambda} J_1 \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} J_1 \ (n \text{ times})$$

induced by the map $\mathcal{O} \otimes_{\mathbf{C}} \mathcal{O} \to \mathcal{O} \otimes_{\mathbf{C}} \mathcal{O} \otimes_{\mathbf{C}} \cdots \otimes_{\mathbf{C}} \mathcal{O}$ $(n+1$ times) $f \otimes g \to f \otimes 1 \otimes 1 \otimes \cdots \otimes 1 \otimes g$, which makes the diagram

$$\begin{tikzcd}
 & & & J_n \arrow[dr, "\lambda_n"] & \\
\mathcal{O} \arrow[r, "D_1"] \arrow[urrr, "D_n"] & J_1 \arrow[r, "D_1"] & J_1 \otimes J_1 \arrow[r, "D_1"] & \cdots\cdots \arrow[r, "D_1"] & J_1 \otimes \cdots\cdots \otimes J_1
\end{tikzcd}$$

commutative. The map $\mathrm{Hom}(J_1 \otimes \cdots \otimes J_1, L) \xrightarrow{\lambda^*} \mathrm{Hom}(J_n, L)$ is surjective, implying the existence of at most one choice for D_n^*, and D_n^* is definable if and only if

(2) $\quad D_1^*(D_1^* \cdots (D_1^*(\phi)) \cdots) = 0$ for $\phi \in \mathrm{Hom}(J_1 \otimes \cdots \otimes J_1, L)$; $\phi(J_n) = 0$.

For $n = 2$ condition (2) is precisely the requirement that d^* define an integrable connection on $\mathrm{Hom}(\Omega^m, L)$. (One has the exact sequence $0 \to J_2 \to J_1 \otimes J_1 \to \Omega^1 \oplus \Omega^2 \to 0$, and the composition $\mathrm{Hom}(\Omega^1, L) \oplus \mathrm{Hom}(\Omega^2, L) \to \mathrm{Hom}(J_1 \otimes J_1, L) \xrightarrow{D^*} \mathrm{Hom}(J_1, L) \xrightarrow{D^*} L$ is $(\alpha, \beta) \to d^*(d^*(\beta))$.) One then establishes (2) by induction on n using the fact that $J_n = (J_2 \otimes_{\mathcal{O}} J_{n-2}) \cap (J_1 \otimes_{\mathcal{O}} J_{n-1})$ as submodules of $J_1 \otimes \cdots \otimes J_1$.

We note finally that D_n^* is suitably $\mathcal{O}$ linear, since the map λ_n^* is $\mathcal{O}$ linear (for both the left and right $\mathcal{O}$ structures) and the maps D_1^* have the appropriate linearity. Thus if $\mathrm{Hom}(\Omega^m, R)$ has an integrable connection, one obtains a special preduality theory.

We have seen that by purely local considerations the only reasonable candidates for duality theories are of the form $M^\cdot \to \mathrm{Hom}_{\mathcal{O}}(M^\cdot, L)$ where $L = \Omega^m \otimes_{\mathbf{C}} V$ with V a local system of rank 1. The *unique* duality theory having the global property $\mathbf{H}^\cdot(X, F(\Omega^\cdot)) = H^\cdot(X, \mathbf{C})$ is the one with $L = \Omega^m$ (since the only rank 1 local system having a global section is trivial). When X is compact, $\mathbf{H}^{2m}(X, F(\Omega^\cdot)) = 0$ for every duality theory except the theory $L = \Omega^m$ for which $\mathbf{H}^{2m}(F(\Omega^\cdot)) = \mathbf{C}$. For this theory one has the following global duality theorem.

Theorem. *Let X be a compact complex manifold or a complete non-singular algebraic variety. Let*

$$\to M^i \xrightarrow{D} M^{i+1} \xrightarrow{D} \cdots$$

be a complex of locally free sheaves with differentials being differential operators. The pth hypercohomology of the complex $\mathrm{Hom}_{\mathcal{O}}(M^\cdot, \Omega^m)$ is dual to the $(2m-p)$th hypercohomology of $M^\cdot$ (as $\mathbf{C}$ vector spaces).

For simplicity we assume the differential operators are all of order $\leqq n$ for some fixed n. The essential ingredients of the proof are

1) $\mathrm{Hom}_{\mathcal{O}}(M^\cdot, \Omega^m) \xrightarrow{\sim} \mathrm{Hom}_{B_n^\cdot}(M^\cdot, \mathrm{Hom}_{\mathcal{O}}(B_n^\cdot, \Omega^m))$ employing the change of rings formula. $M^\cdot$ is a $B_n^\cdot$ module by the natural inclusion $B_n^\cdot \to A_n$. Under the identification (1) the differential D^* in this complex defined by duality may be calculated to be

$$(D^*\Phi)(m) = (-1)^p d^*(\Phi(m)) + (-1^{p+1}\Phi(D(m)) \quad \text{for } m \in M^p$$

where d^* is the differential in $\mathrm{Hom}_{\mathcal{O}}(B_n^\cdot, \Omega^m)$ (also defined by duality).

2) Thus there is a natural pairing of complexes

$$M^\cdot \times (\mathrm{Hom}_{\mathcal{O}} M^\cdot, \Omega^m) \to \mathrm{Hom}_{\mathcal{O}}(B_n^\cdot, \Omega^m)$$

(in terms of the identification in (1), $\alpha \times \Phi \to \Phi(\alpha)$).

3) The pairing in 2) gives rise to a natural pairing of the E_1 spectral sequences of hypercohomology

$$\begin{array}{ccccc}
\mathbf{H}_p(M^\cdot) & \times & \mathbf{H}^{2m-p}(\mathrm{Hom}\, M^\cdot, \Omega^m)) & \to & \mathbf{H}^{2m}(\mathrm{Hom}(B_n, \Omega^m)) \\
\Uparrow & & \Uparrow & & \uparrow \\
H^a(M^b) & \times & \mathrm{H}^{m-a}(\mathrm{Hom}\,(M^b, \Omega^m)) & \to & H^m(\Omega^m) = \mathrm{H}^m(\mathrm{Hom}(B_n^\cdot, \Omega^m)).
\end{array}$$

(cf. [7]).

4) The pairing is perfect at level E_1 (Serre duality).

5) $\mathbf{H}^\cdot(\mathrm{Hom}(B_n, \Omega^m)) \xrightarrow{\sim} H^\cdot(X, \mathbf{C})$ and in particular $\mathbf{H}^{2m}(\mathrm{Hom}(B_n, \Omega^m) \xrightarrow{\sim} \mathbf{C}$.

6) From 5) it follows that the edge morphism $H^m(\Omega^m) \to H^{2m}(\mathrm{Hom}(B_n, \Omega^m)$ is an isomorphism and that all differentials in this spectral sequence mapping into $H^m(\Omega^m)$ are zero. Thus as in [7], §5, the differentials in the paired spectral sequences are dual.

7) The duality of the abutments of the spectral sequences follows from the regularity of the spectral sequences as in [7].

One can extend the duality theory to handle complexes $M^\cdot$ in which the M^i are only assumed to be coherent by introducing the notion of hyperext essentially following the program of [7]. Namely if $E^{\cdot}$ and $F^{\cdot}$ are $C_n^{\cdot}$ modules, then the graded group $\mathrm{Hom}^{\cdot}_{B_n}(E^{\cdot}, F^{\cdot})$ has a natural differential defined by

$$(D^*\Phi)(e) = (-1)^p D_F(\Phi(e)) + (-1)^{p+1}\Phi(D_E(e)) \quad \text{for } e \in E^p.$$

Taking $Q^{\cdot}$ a C_n injective resolution of $F^{\cdot}$ (i.e., imbedding $0 \to F^{\cdot} \xrightarrow{i} Q^{\cdot}$ with $Q^{\cdot}$ a C_n injective and i inducing an isomorphism of cohomology sheaves), one defines $Ext^{\cdot}(E^{\cdot}, F^{\cdot})$ to be the cohomology of the complex $\mathrm{Hom}^{\cdot}_{B_n}(E, Q)$ (it is independent of $Q^{\cdot}$). Then $\mathbf{H}^p(E^{\cdot})$ is dual to $\mathrm{Ext}^{2m-p}(E^{\cdot}, \mathrm{Hom}(B_n, \Omega^m))$ under the natural Yoneda pairing (cf. [7], §§4, 5).

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